AIAA JOURNAL
Vol. 36, No. 9, September 1998

Design Sensitivity Metric for Structural Dynamic Response

Hector A. Jensen*
Federico Santa Maria University, Valparaiso, Chile
and
Abdon E. Sepulveda†
University of California, Los Angeles, Los Angeles, California 90095

An approach to study the sensitivity of dynamic responses for structural systems with respect to a set of design parameters is presented. The sensitivity is evaluated by considering the behavior of the system response when the design parameters vary within a given region of the design space. The sensitivity is computed by means of approximate responses, which are evaluated using approximation concepts. The approximation is based on modal analysis, and it is valid for general underdamped linear systems. Intermediate design variable and intermediate response quantity concepts are used to enhance the accuracy of the approximation. In this approach, modal energies and mode shapes are chosen as intermediate response quantities, and they are approximated in terms of selected intermediate system parameters. The approximation of these quantities requires a standard eigenvalue and eigenvector sensitivity analysis. Numerical results that illustrate the usefulness and effectiveness of the method are presented. Great insight into the behavior of the system can be gained using this methodology.

Nomenclature

$[C]$	= damping matrix
$\{f\}$	= vector of intermediate response quantities
$[K]$	= stiffness matrix
$[M]$	= mass matrix
$\{p\}$	= excitation vector
$\{q\}$	= state-space vector
$\{R\}$	= general system response
S	= sensitivity matrix
S_r, T_r, U_r	= modal energies
t	= time variable
t_0	= initial time
$\{u\}$, $\{\dot{u}\}$, $\{\ddot{u}\}$	= vectors of dynamic displacements, velocities, and accelerations, respectively
$\{u_0\}$, $\{\dot{u}_0\}$	= initial displacements and velocities
$\{x\}$	= vector of intermediate system parameters
$\{y\}$	= vector of design variables
$\{y_0\}$	= vector of baseline design variables
$\eta_r(t)$	= modal participation coefficient
λ_r	= eigenvalue
μ	= measure of the range of variation of a design variable
ξ_r	= modal damping coefficient
Υ	= sensitivity metric
$\Upsilon^{\max}$	= sensitivity metric for peak response
$\{\phi_{pr}\}$	= position part of the right eigenvector
$\{\phi_r\}$	= right eigenvector
$\{\phi_{vr}\}$	= velocity part of the right eigenvector
$\{\chi_{pr}\}$	= position part of the left eigenvector
$\{\chi_r\}$	= left eigenvector
$\{\chi_{vr}\}$	= velocity part of the left eigenvector
ω_r	= natural frequency
$\max_t(\cdot)$	= maximum value with respect to t
$\mathrm{Re}(\cdot)$	= real part
$\bar{(\cdot)}$	= complex conjugate
$\tilde{(\cdot)}$	= approximate quantity

Introduction

SENSITIVITY analysis of structural systems plays a critical role in the design and analysis of structures, as well as in numerical optimization, reliability analyses, and identification studies. The basic concepts of sensitivity analysis of structural response are well documented in a number of publications.[1–3] In general, the sensitivity of the system response is evaluated by partial derivatives of some response functions with respect to the system parameters. Methods for computing partial derivatives of structural responses include finite difference methods, direct differentiation methods, and adjoint methods. All of these methods consider the variability in the system response due to local variation of the design parameters, that is, they establish a measure of the way in which the response varies with changes in the parameters in the neighborhood of their nominal values.

Much work has been performed in the area of structural design sensitivity analysis. Sensitivities have been derived for a number of systems with respect to a wide range of parameters. For example, sensitivity with respect to material properties[4,5]; sectional parameters that describe beams, plates, and shells[6,7]; and shape parameters that describe the body's geometry.[8,9] This information is used to predict how the response function value varies for small perturbations in the model parameters without performing a reanalysis. Thus, the sensitivities offer an efficient means of predicting the local performance of modified models. One important application of this methodology is in the area of structural optimization with applications to numerous structural systems.[10–12] Sensitivities have also been derived for other classes of problems, such as nonlinear structural systems, eigenvalue and frequency responses, elastodynamic systems, thermal systems, fluid dynamic systems, rigid-body mechanics, and general field problems.[13–18]

The objective of this paper is to introduce a new sensitivity metric in the context of structural dynamic response. The method is based on modal analysis, and it is valid for general underdamped structures, which are the cases of primary interest in this study. The sensitivity of the system is evaluated by means of approximate responses. The use of approximation concepts with the use of intermediate response quantity and intermediate design variable concepts are considered for the efficient evaluation of the approximations.[19,20] The use of approximation concepts is essential because the direct evaluation of response functions for complex systems in the area of structural dynamics is prohibitively expensive in terms of computational resources. Approximate responses are used to define coefficients of sensitivity that measure the variability of the system response when the design parameters vary within a given region of the design space.

Received Jan. 28, 1998; revision received May 17, 1998; accepted for publication May 20, 1998.

*Associate Professor, Department of Civil Engineering, Av. Los Placeres 401, V Region, Casilla 110-V.

†Assistant Professor, Mechanical, Aerospace and Nuclear Engineering Department, 405 Hilgard Avenue. Senior Member AIAA.

This type of approach takes into account that some system parameters are difficult to determine and are usually estimated with some margin of error. If the sensitivity is obtained without quantifying the effects of these errors in the modeling of the system parameters, then the results can lead to misleading conclusions. The proposed method can also identify the more influential design variables on the behavior of the system and the less influential variables in a particular domain of the design space. This sensitivity information is also useful in the prediction of the system response for large perturbations in the design variables and system parameters without performing a costly reanalysis.

First, the dynamic response of a general underdamped linear system based on the modal solution of the equations of motion is presented. Then, approximation concepts are introduced for an efficient numerical implementation of the method. Next, a sensitivity metric is defined in order to perform the proposed sensitivity analysis. Finally, some example problems are considered to illustrate the newly developed approach.

Dynamic Response

The general matrix equation of motion for an n-degree-of-freedom linear structure is given by

$$[M]\{\ddot{u}\} + [C]\{\dot{u}\} + [K]\{u\} = \{p\} \tag{1}$$

where $[M]$, $[C]$, and $[K]$ are the $n \times n$ mass, damping, and stiffness matrices, respectively; $\{u\}$ is the vector of dynamic displacements; and $\{p\}$ is the excitation vector.

In this formulation, an approximation for the vector of dynamic displacements $\{u\}$ as an explicit function of a set of design parameters is required. Such approximation can be based on the direct solution of Eq. (1) or by modal analysis. In the direct solution approach, an equation for the sensitivity of the dynamic displacement vector $\{u\}$ is first obtained by differentiating Eq. (1). Then, the equation is solved using some numerical integration technique. With this sensitivity information, the displacement vector is approximated using first- or second-order Taylor series. The system responses obtained with this approach are, in general, poorly approximated, especially near resonance conditions.[19,21,22] In view of these difficulties, the equations of motion (1) are solved by using modal analysis. To consider the general case, it is assumed that Eq. (1) cannot be uncoupled by means of natural modes because this can be done only under certain conditions for the damping matrix $[C]$, e.g., proportional damping. The general case to be presented simplifies to the usual second-order differential equation for the modal participation coefficients, when the system is decoupled by natural modes.

The problem is treated by transforming the set of n differential equations of second order into a set of $2n$ differential equations of first order. To this end, the state-space variables are defined as

$$\{q\} = \begin{Bmatrix} \{\dot{u}\} \\ \{u\} \end{Bmatrix} \tag{2}$$

which, introduced into Eq. (1), leads to the equations of motion in first-order form, namely,

$$[M^*]\{\dot{q}\} + [K^*]\{q\} = \{p\}^* \tag{3}$$

where

$$[M^*] = \begin{pmatrix} [0] & [M] \\ [M] & [C] \end{pmatrix}, \qquad [K^*] = \begin{pmatrix} -[M] & [0] \\ [0] & [K] \end{pmatrix}$$
$$\{p\}^* = \begin{Bmatrix} \{0\} \\ \{p\} \end{Bmatrix} \tag{4}$$

In the modal approach, it is assumed that the dynamic state-space response can be represented as a linear combination of complex mode shapes of the form

$$\{q\} = \sum_{r=1}^{2n} \{\phi\}_r \eta_r(t) \tag{5}$$

where $\eta_r(t)$, $r = 1, \ldots, 2n$ are the modal participation coefficients and $\{\phi_r\}$, $r = 1, \ldots, 2n$ are the complex right eigenvectors corresponding to Eq. (3), that is, they are the solution of the right eigenproblem

$$([K^*] + \lambda[M^*])\{\phi\} = \{0\} \tag{6}$$

In this formulation, the left eigenproblem associated to Eq. (3) is also needed, that is,

$$\{\chi\}^t([K^*] + \lambda[M^*]) = \{0\}^t \tag{7}$$

where $\{\chi\}$ is the left eigenvector. For the cases of underdamped systems, the solution of these eigenproblems leads to $2n$ complex conjugate eigenvalues $\lambda_1, \ldots, \lambda_{2n}$, and the associated complex conjugate right and left eigenvectors $\{\phi\}_1, \ldots, \{\phi\}_{2n}$ and $\{\chi\}_1, \ldots, \{\chi\}_{2n}$. These eigenvectors are orthogonal with respect to the $[M^*]$ and $[K^*]$ matrices, i.e.,

$$\{\chi\}_i^t[M^*]\{\phi\}_j = 0, \qquad i \neq j, \qquad i, j = 1, \ldots, 2n \tag{8}$$

$$\{\chi\}_i^t[K^*]\{\phi\}_j = 0, \qquad i \neq j, \qquad i, j = 1, \ldots, 2n \tag{9}$$

For convenience, the following numbering for the eigenvalues and eigenvectors is assumed: $\lambda_{n+i} = \bar{\lambda}_i$, $\{\phi\}_{n+i} = \{\bar{\phi}\}_i$, and $\{\chi\}_{n+i} = \{\bar{\chi}\}_i$, $i = 1, \ldots, n$, where $\bar{\lambda}_i$, $\{\bar{\phi}\}_i$, and $\{\bar{\chi}\}_i$ denote the complex conjugates of λ_i, $\{\phi\}_i$, and $\{\chi\}_i$, respectively. The modes are ordered in ascending order of the imaginary parts (damped frequencies). Substituting Eq. (5) in Eq. (3), premultiplying by the complex left eigenvector $\{\chi\}_r^t$, and using the orthogonality of the left and right eigenvectors leads to

$$T_r^* \dot{\eta}_r(t) + U_r^* \eta_r(t) = \{\chi\}_r^t \{p\}^* \tag{10}$$

where

$$T_r^* = \{\chi\}_r^t[M^*]\{\phi\}_r, \qquad U_r^* = \{\chi\}_r^t[K^*]\{\phi\}_r \tag{11}$$

From the definition of the right and left eigenproblems and Eq. (11), it is easily shown that the corresponding eigenvalue λ_r satisfies

$$\lambda_r = -U_r^* / T_r^* \tag{12}$$

and, therefore, Eq. (10) can be written as

$$\dot{\eta}_r(t) - \lambda_r \eta_r(t) = \frac{\{\chi\}_r^t \{p\}^*}{T_r^*}, \qquad r = 1, \ldots, 2n \tag{13}$$

If the complex eigenvectors are partitioned in velocity and position parts as

$$\{\phi\}_r = \begin{Bmatrix} \{\phi\}_{vr} \\ \{\phi\}_{pr} \end{Bmatrix} \tag{14}$$

$$\{\chi\}_r = \begin{Bmatrix} \{\chi\}_{vr} \\ \{\chi\}_{pr} \end{Bmatrix} \tag{15}$$

in which the velocity and position parts are related from Eqs. (4) and (6) as

$$\{\phi\}_{vr} = \lambda_r \{\phi\}_{pr} \tag{16}$$

$$\{\chi\}_{vr} = \lambda_r \{\chi\}_{pr} \tag{17}$$

then the following identities are obtained:

$$T_r^* = 2\lambda_r T_r + S_r, \qquad U_r^* = -\lambda_r^2 T_r + U_r \tag{18}$$

where T_r, U_r, and S_r are given by

$$T_r = \{\chi\}_{pr}^t[M]\{\phi\}_{pr}, \qquad U_r = \{\chi\}_{pr}^t[K]\{\phi\}_{pr}$$
$$S_r = \{\chi\}_{pr}^t[C]\{\phi\}_{pr} \tag{19}$$

and where $\{\phi\}_{pr}$ and $\{\chi\}_{pr}$ are the position parts of the right and left eigenvector, respectively. In the context of this formulation, the terms T_r and U_r are defined as modal energies. This is due to the similarity of their expressions with the definition of kinetic and potential energy for undamped systems. For extension, the term S_r is also called a modal energy.

Introducing the definition of T_r^* and $\{p\}^*$ in Eq. (13), the differential equation for the modal participation coefficients can be written as

$$\dot{\eta}_r(t) - \lambda_r \eta_r(t) = \frac{\{\chi\}_{pr}^t \{p\}}{(2\lambda_r T_r + S_r)}, \qquad r = 1, \ldots, 2n \tag{20}$$

where the coefficients appear in complex conjugate pairs, that is, $\eta_{n+r}(t) = \bar{\eta}_r(t)$, $r = 1, \ldots, n$, and where $\bar{\eta}_r(t)$ is the complex conjugate of $\eta_r(t)$.

The initial conditions for Eq. (20) can be obtained directly from Eq. (5). Evaluating this equation at the initial time $t = t_0$ and premultiplying by $\{\chi\}_r^t[M^*]$ leads to

$$\eta_r(t_0) = \frac{\{\chi\}_r^t[M^*]\{q(t_0)\}}{T_r^*} \tag{21}$$

or using Eqs. (2), (15), (17), and (18),

$$\eta_r(t_0) = \frac{\lambda_r\{\chi\}_{pr}^t[M]\{u_0\} + \{\chi\}_{pr}^t[C]\{u_0\} + \{\chi\}_{pr}^t[M]\{\dot{u}_0\}}{2\lambda_r T_r + S_r} \tag{22}$$

where $\{u_0\}$ and $\{\dot{u}_0\}$ are the initial displacements and velocities.

Finally, defining the complex quantities

$$TV_r = \{\chi\}_{pr}^t[M]\{\dot{u}_0\}, \qquad TD_r = \{\chi\}_{pr}^t[M]\{u_0\}$$
$$SD_r = \{\chi\}_{pr}^t[C]\{u_0\} \tag{23}$$

the initial condition for the modal participation coefficients can be written as

$$\eta_r(t_0) = \frac{\lambda_r TD_r + SD_r + TV_r}{2\lambda_r T_r + S_r}, \qquad r = 1, \ldots, 2n \tag{24}$$

and the general solution of Eq. (20), with initial condition at $t = t_0$, is given by

$$\eta_r(t) = e^{\lambda_r(t - t_0)}\eta_r(t_0) + \int_{t_0}^{t} \frac{e^{\lambda_r(t-\xi)}\{\chi\}_{pr}^t\{p(\xi)\}}{(2\lambda_r T_r + S_r)}\,\mathrm{d}\xi$$
$$r = 1, \ldots, 2n \tag{25}$$

If general load vectors are considered, numerical integration of Eq. (25) is unavoidable to obtain $\eta_r(t)$. To alleviate this computational burden and obtain closed-form solutions, it is assumed that the load vector is represented by a piecewise linear function in time. This is the case for a wide range of applications, including the cases where the signal input is in the form of a set of discrete numbers that represents the sample of the signal at different equally spaced values. It is noted, however, that the formulation presented here is also valid for representations of the load vector other than the one considered in this study. In that case, the difference is the computational effort required to solve Eq. (25).

Under the just stated assumption, the analysis time interval $[t_0, t_f]$ is subdivided such that $t_0 < t_1 < \cdots < t_m = t_f$, and the load vector is specified at each time t_i, $\{p(t_i)\}$, $i = 0, 1, \ldots, m$. Thus, for the interval $[t_i, t_{i+1}]$ the forcing vector is given by

$$\{p(t)\} = \{a_i\} + \{b_i\}t \tag{26}$$

where

$$\{a_i\} = \frac{\{p(t_i)\}t_{i+1} - \{p(t_{i+1})\}t_i}{t_{i+1} - t_i}, \qquad \{b_i\} = \frac{\{p(t_{i+1})\} - \{p(t_i)\}}{t_{i+1} - t_i} \tag{27}$$

The solution for $\eta_r(t)$ in the interval $[t_i, t_{i+1}]$, denoted by $\eta_r^i(t)$, is given by

$$\eta_r^i(t) = e^{\lambda_r(t - t_i)}\left[\eta_r^{i-1}(t_i) - \alpha_r^i - \beta_r^i t_i\right] + \alpha_r^i + \beta_r^i t$$
$$t \in [t_i, t_{i+1}] \tag{28}$$

where

$$\alpha_r^i = -\frac{\{\chi\}_{pr}^t(\{b_i\} - \lambda_r\{a_i\})}{\lambda_r^2(2\lambda_r T_r + S_r)}, \qquad \beta_r^i = -\frac{\{\chi\}_{pr}^t\{b_i\}}{\lambda_r(2\lambda_r T_r + S_r)} \tag{29}$$

Equation (28) gives a recursive formula to evaluate $\eta_r^i(t)$ for each interval $[t_i, t_{i+1}]$. In this manner, the participation coefficients $\eta_r^i(t)$, $i = 1, \ldots, m$, are obtained directly from Eqs. (28) and (29), together with the initial condition $\eta_r(t_0)$.

Finally, from Eq. (5) and using the fact that the complex eigenvectors appear in complex conjugate pairs, the vector of dynamic displacements is given by

$$\{u(t)\} = 2\,\mathrm{Re}\left[\sum_{r=1}^{n}\{\phi\}_{pr}\eta_r(t)\right] \tag{30}$$

where $\eta_r(t)$ is given by Eq. (28) and Re denotes the real part of a complex number.

It is noted that, for the cases in which the equations of motion (1) can be uncoupled by natural modes, Eq. (30) becomes

$$\{u(t)\} = \sum_{r=1}^{n}\{\phi\}_r\eta_r(t) \tag{31}$$

where $\{\phi\}_r$ is the rth natural mode shape and $\eta_r(t)$ is the modal participation coefficient satisfying

$$\ddot{\eta}_r(t) + 2\xi_r\omega_r\dot{\eta}_r(t) + \omega_r^2\eta_r(t) = \frac{\{\phi\}_r^t\{p\}}{T_r}, \qquad r = 1, \ldots, n \tag{32}$$

where ω_r is the rth natural frequency, ξ_r is the modal damping coefficient for the rth mode, $T_r = \{\phi\}_r^t[M]\{\phi\}_r$, and all other terms are as defined earlier.

Approximation Concepts

It is clear, from Eqs. (25) and (30), that the response of the system $\{u\}$ depends on its spectral properties, that is, $\{\phi\}_r$, $\{\chi\}_r$, and λ_r, $r = 1, \ldots, 2n$. At the same time, these properties are implicit nonlinear functions of the vector of design variables or structural parameters, which are denoted by $\{y\}$ (y_j, $j = 1, \ldots, l$).

A general system response $\{R(t, \{y\})\}$ (R_i, $i = 1, \ldots, m$) can be written as

$$\{R(t, \{y\})\} = \{H(t, \{f(\{x(\{y\})\})\}, \{x(\{y\})\}, \{y\})\} \tag{33}$$

where $\{f\}$ (f_i, $i \in I$) are intermediate response quantities, $\{x\}$ (x_j, $j \in J$) are intermediate system parameters, and I and J are sets of indices. In Eq. (33) it is assumed that 1) $\{H\}$ is explicit in $\{f\}$, $\{x\}$, $\{y\}$, and t; 2) f_i, $i \in I$, are implicit functions of $\{x\}$; and 3) x_j, $j \in J$, are explicit functions of $\{y\}$.

The evaluation of the intermediate response quantities (f_i, $i \in I$) is, in general, very costly in terms of computational resources because these functions are available only in an algorithmic or numerical way, for instance, by means of a finite element model. In this approach, analytical approximations of the intermediate response quantities are used, and they are constructed by approximating the functions f_i, $i \in I$, explicitly in terms of the intermediate design variables $\{x\}$, to give $\tilde{f}_i$, $i \in I$. Once these approximations have been obtained, the system response $\{R(t, \{y\})\}$ can be approximated and written explicitly in terms of the set of original design variables or structural parameters $\{y\}$ due to the explicitness of the function $\{H\}$. In this approach, the modal energies T_r, S_r, and U_r and the position parts of the right and left eigenvectors $\{\phi\}_{pr}$ and $\{\chi\}_{pr}$ are chosen as intermediate response quantities.

To illustrate these concepts, consider the case of $\{R\} = \{u\}$. From Eq. (30) it is clear that

$$\{H\} = 2\,\mathrm{Re}\left[\sum_{r=1}^{n}\{\phi\}_{pr}\eta_r(t)\right]$$

At the same time, Eq. (25) indicates that the modal participation coefficients depend on the modal energies and the left eigenvectors. Therefore, these quantities together with the right eigenvectors can be considered as intermediate response quantities. These responses are implicit functions of intermediate system parameters such as cross-sectional properties, axial and bending rigidity, etc. Finally, these intermediate system parameters are explicit functions of the actual design variables or structural parameters, for example, cross-sectional dimensions.

In this formulation, the intermediate response quantities are approximated locally in Taylor series with respect to selected intermediate system parameters as

$$\tilde{T}_r = T_{r0} + \sum_j \frac{\partial T_r(\{x_0\})}{\partial x_j}(x_j - x_{j0}) + \sum_{ij} \frac{\partial^2 T_r(\{x_0\})}{\partial x_i \partial x_j}(x_i - x_{i0})(x_j - x_{j0}) + \cdots \tag{34}$$

$$\tilde{S}_r = S_{r0} + \sum_j \frac{\partial S_r(\{x_0\})}{\partial x_j}(x_j - x_{j0}) + \sum_{ij} \frac{\partial^2 S_r(\{x_0\})}{\partial x_i \partial x_j}(x_i - x_{i0})(x_j - x_{j0}) + \cdots \tag{35}$$

$$\tilde{U}_r = U_{r0} + \sum_j \frac{\partial U_r(\{x_0\})}{\partial x_j}(x_j - x_{j0}) + \sum_{ij} \frac{\partial^2 U_r(\{x_0\})}{\partial x_i \partial x_j}(x_i - x_{i0})(x_j - x_{j0}) + \cdots \tag{36}$$

$$\{\tilde{\phi}\}_{pr} = \{\phi\}_{pr0} + \sum_j \frac{\partial\{\phi(\{x_0\})\}_{pr}}{\partial x_j}(x_j - x_{j0}) + \sum_{ij} \frac{\partial^2\{\phi(\{x_0\})\}_{pr}}{\partial x_i \partial x_j}(x_i - x_{i0})(x_j - x_{j0}) + \cdots \tag{37}$$

$$\{\tilde{\chi}\}_{pr} = \{\chi\}_{pr0} + \sum_j \frac{\partial\{\chi(\{x_0\})\}_{pr}}{\partial x_j}(x_j - x_{j0}) + \sum_{ij} \frac{\partial^2\{\chi(\{x_0\})\}_{pr}}{\partial x_i \partial x_j}(x_i - x_{i0})(x_j - x_{j0}) + \cdots \tag{38}$$

where $T_{r0} = T_r(\{x_0\})$, $S_{r0} = S_r(\{x_0\})$, $U_{r0} = U_r(\{x_0\})$, $\{\phi\}_{pr0} = \{\phi(\{x_0\})\}_{pr}$, $\{\chi\}_{pr0} = \{\chi(\{x_0\})\}_{pr}$, $\{x_0\} = \{x(\{y_0\})\}$, and $\{y_0\}$ corresponds to the vector of design variables $\{y\}$ when the values of the components are equal to their base design values. One simplification of these formulas, widely used in structural synthesis, is to assume that the mode shapes are invariant. This makes derivative calculation very inexpensive from a computational point of view. However, in this formulation, the variability of the mode shapes is considered explicitly in the approximations to increase the size of the design space where the approximations yield reasonable results.

In general, the range of validity of the approximations depends on the degree of nonlinearity of the intermediate responses as a function of the intermediate design variables. The approximations of the intermediate response functions in terms of selected intermediate variables are used to retain the explicit nonlinear dependence of the system response on the intermediate response quantities and the explicit relation between the intermediate design variables and the actual design variables. Based on the numerical examples performed by the authors, variabilities up to 40% of the design variables with respect to their nominal values can be considered without significant loss of accuracy in the approximations.

The evaluation of the partial derivatives used in the approximations of the intermediate response quantities [Eqs. (34–38)] requires a standard eigenvalue–eigenvector sensitivity analysis at the base design, that is, eigenvalue and eigenvector derivatives at $\{x_0\}$. The eigenvalue derivative with respect to an intermediate design parameter is given by[13]

$$\frac{\partial \lambda_r}{\partial x_j} = -\left[\{\chi\}_r^t \left(\frac{\partial[K^*]}{\partial x_j} + \lambda_r \frac{\partial[M^*]}{\partial x_j}\right)\{\phi\}_r\right] \Big/ \{\chi\}_r^t[M^*]\{\phi\}_r \tag{39}$$

Using the definition of the matrices $[K^*]$ and $[M^*]$ and Eqs. (14) and (15), it follows that

$$\{\chi\}_r^t \frac{\partial[K^*]}{\partial x_j}\{\phi\}_r = -\lambda_r^2 \{\chi\}_{pr}^t \frac{\partial[M]}{\partial x_j}\{\phi\}_{pr} + \{\chi\}_{pr}^t \frac{\partial[K]}{\partial x_j}\{\phi\}_{pr} \tag{40}$$

$$\{\chi\}_r^t \frac{\partial[M^*]}{\partial x_j}\{\phi\}_r = 2\lambda_r \{\chi\}_{pr}^t \frac{\partial[M]}{\partial x_j}\{\phi\}_{pr} + \{\chi\}_{pr}^t \frac{\partial[C]}{\partial x_j}\{\phi\}_{pr} \tag{41}$$

$$\{\chi\}_r^t[M^*]\{\phi\}_r = 2\lambda_r \{\chi\}_{pr}^t[M]\{\phi\}_{pr} + \{\chi\}_{pr}^t[C]\{\phi\}_{pr} \tag{42}$$

On the other hand, the derivative of the right eigenvector with respect to an intermediate design variable can be written as[13]

$$\frac{\partial\{\phi\}_{pr}}{\partial x_j} = \{\varphi\}_{pr} + c_r\{\phi\}_{pr} \tag{43}$$

where

$$c_r = -\mathrm{Re}\begin{Bmatrix} \bar{\lambda}_r\{\bar{\phi}\}_{pr} \\ \{\bar{\phi}\}_{pr} \end{Bmatrix}^t \begin{Bmatrix} \{\varphi\}_{vr} \\ \{\varphi\}_{pr} \end{Bmatrix} \tag{44}$$

and where

$$\{\varphi\}_r = \begin{Bmatrix} \{\varphi\}_{vr} \\ \{\varphi\}_{pr} \end{Bmatrix} \tag{45}$$

is a particular solution of the problem

$$\big([K^*] + \lambda_r[M^*]\big)\{\varphi\}_r = -\left(\frac{\partial[K^*]}{\partial x_j} + \lambda_r \frac{\partial[M^*]}{\partial x_j} + \frac{\partial \lambda_r}{\partial x_j}[M^*]\right)\{\phi\}_r \tag{46}$$

The difficulty in determining the vector $\{\varphi\}_r$ is that the matrix on the left-hand side of Eq. (46) is at most of rank $2n - 1$ and cannot be inverted. In this formulation, the simplified calculation of a particular solution proposed by Nelson[13] was implemented in the numerical examples considered. A similar derivation can be performed for the derivative of the left eigenvectors with respect to intermediate design variables. If second-order expansions are considered for the intermediate response quantities [Eqs. (34–38)], then a second-order eigenvalue–eigenvector sensitivity analysis is required.

Next, introducing the approximations given by Eqs. (34–38) in Eq. (25) gives

$$\tilde{\eta}_r(t) = e^{\tilde{\lambda}_r(t-t_0)}\tilde{\eta}_r(t_0) + \int_{t_0}^{t} \frac{e^{\tilde{\lambda}_r(t-\xi)}\{\tilde{\chi}\}_{pr}^t\{p(\xi)\}}{(2\tilde{\lambda}_r\tilde{T}_r + \tilde{S}_r)}\,\mathrm{d}\xi \qquad r = 1, \ldots, 2n \tag{47}$$

It is noted that the eigenvalue λ_r can be written in terms of the modal energies directly from Eqs. (12) and (18). Substituting Eq. (18) into Eq. (12) gives

$$\lambda_r^2 T_r + \lambda_r S_r + U_r = 0 \tag{48}$$

and, therefore, the approximate eigenvalue can be expressed as

$$\tilde{\lambda}_r = \frac{-\tilde{S}_r \pm \sqrt{\tilde{S}_r^2 - 4\tilde{T}_r\tilde{U}_r}}{2\tilde{T}_r}, \qquad r = 1, \ldots, 2n \tag{49}$$

Equation (49) gives $2n$ different solutions for the approximate eigenvalues, which are complex conjugates. To approximate $\eta_r(t_0)$, the quantities TD_r, SD_r, and TV_r in Eq. (23) are chosen as additional intermediate response quantities. These intermediate responses are approximated in a similar manner as T_r, U_r, and S_r. The initial conditions $\{u_0\}$ and $\{\dot{u}_0\}$ are considered constant but can also be considered as functions of the design variables. Then, the approximation for the initial condition $\tilde{\eta}_r(t_0)$ is given by

$$\tilde{\eta}_r(t_0) = \frac{\tilde{\lambda}_r \widetilde{TD}_r + \widetilde{SD}_r + \widetilde{TV}_r}{2\tilde{\lambda}_r \tilde{T}_r + \tilde{S}_r} \tag{50}$$

In this way, the approximation for the transient dynamic displacements is constructed using Eq. (30) with a truncated set of modes and the approximate modal participation coefficients given by Eq. (47). Then

$$\{\tilde{u}(t)\} = 2\,\mathrm{Re}\left[\sum_{r=1}^{N} \{\tilde{\phi}\}_{pr} \tilde{\eta}_r(t)\right] \tag{51}$$

where N is the retained number of modes.

In summary, the quantities T_r, S_r, U_r, TV_r, TD_r, TD_r, $\{\phi\}_{pr}$, and $\{\chi\}_{pr}$ for all of the retained modes are chosen as intermediate response quantities, and they are approximated in terms of appropriate intermediate design variables. The approximation of these quantities requires a standard eigenvalue and eigenvector sensitivity analysis. It is noted that the transient dynamic displacement given by Eq. (51) is an explicit function of the set of original design variables or structural parameters. Therefore, a general system response $\{R(t, \{y\})\}$ can be approximated and written explicitly in terms of the vector of system parameters $\{y\}$ as

$$\{\tilde{R}(t, \{y\})\} = \{H(t, \{\tilde{f}(\{x(\{y\})\})\}, \{x(\{y\})\}, \{y\})\} \tag{52}$$

Because now the general system response is approximate, the evaluation of response functions for complex systems is feasible from a computational point of view due to the explicitness of the function $\{\tilde{R}(t, \{y\})\}$.

Sensitivity Measures

The approximation of the general system response $\{R(t, \{y\})\}$ $(R_i, i = 1, \ldots, m)$ is completely defined by Eq. (52). This approximation is written explicitly in terms of the vector of design variables or structural parameters. Thus, $\{\tilde{R}(t, \{y\})\}$ $(\tilde{R}_i, i = 1, \ldots, m)$ represents an approximation of the response $\{R(t, \{y\})\}$. Using this characterization, the response of a complex system corresponding to a given set of values of its parameters can be obtained directly and efficiently by evaluating Eq. (52) on such a set of values. Thus, the behavior of the response for large perturbations in the system parameters can be evaluated without performing a reanalysis. In the same manner, contour plots can be derived directly from that characterization. The approximation given by Eq. (52) is also well suited for computing approximate peak value responses. The maximum transient response for a given set of values of the system parameters can be obtained directly by evaluating in time the analytical approximation of the response on such a set of parameters and then choosing its maximum value. In short,

$$\tilde{R}_{i,\max}(\{y\}) = \max_t[\tilde{R}_i(t, \{y\})], \qquad i = 1, \ldots, m \tag{53}$$

where $\tilde{R}_{i,\max}(\{y\})$ is the peak value response.

Another application of this formulation is in connection with sensitivity analysis. In this formulation, the sensitivity is evaluated by considering the behavior of the system response when the parameters belong to a given region of the design space. For example, the sensitivity of the response R_i with respect to a system parameter y_j can be measured by the dispersion of the response about the base or nominal response through the coefficient of sensitivity or sensitivity metric

$$\Upsilon_{R_i, y_j}(t) = \frac{\sqrt{[1/\mu(y_j)] \int_{y_j} [\tilde{R}_i(t, \{y\}) - \bar{R}_i(t)]^2 \, \mathrm{d}y_j}}{\max_t[\bar{R}_i(t)]} \tag{54}$$

where $\mu(y_j)$ is the measure of the range of variation of the design variable y_j, $\bar{R}_i(t)$ is the baseline response, $y_k = \{y_0\}_k, k \neq j$, and $\max_t[\bar{R}_i(t)]$ is the maximum baseline response in time. In this context, the baseline response corresponds to the system response when the values of the parameters are equal to their base design values, i.e., $R_i(t, \{y_0\})$. This coefficient can be evaluated numerically by using the characterization of the approximate response $\tilde{R}_i(t, \{y\})$. The coefficient of Eq. (54) can also be used to define a sensitivity or coupling matrix as

$$S(t) = \begin{bmatrix} \Upsilon_{R_1, y_1}(t) & \ldots & \Upsilon_{R_m, y_1}(t) \\ \vdots & \ddots & \vdots \\ \Upsilon_{R_1, y_l}(t) & \ldots & \Upsilon_{R_m, y_l}(t) \end{bmatrix} \tag{55}$$

where R_i, $i = 1, \ldots, m$, are, as before, the system responses, and y_i, $i = 1, \ldots, l$, the design variables or structural parameters. This matrix determines the degree of functional coupling in the set of design variables with respect to different response functions. Coefficients of sensitivity and coupling matrices can also be defined for peak responses. In this case, the coefficient of sensitivity or sensitivity metric is defined as

$$\Upsilon^{\max}_{R_i, y_j} = \frac{\sqrt{[1/\mu(y_j)] \int_{y_j} (\max_t[\tilde{R}_i(t, \{y\})] - \max_t[\bar{R}_i(t)])^2 \, \mathrm{d}y_j}}{\max_t[\bar{R}_i(t)]} \tag{56}$$

where the peak response $\max_t[\tilde{R}_i(t, \{y\})]$ can be identified directly by evaluating in time the approximate response and then choosing its maximum value. This evaluation is feasible from a computational point of view due to the explicitness of $\tilde{R}_i(t, \{y\})$ with respect to the system parameters $\{y\}$. The corresponding coupling matrix for peak responses is given by

$$S = \begin{bmatrix} \Upsilon^{\max}_{R_1, y_1} & \ldots & \Upsilon^{\max}_{R_m, y_1} \\ \vdots & \ddots & \vdots \\ \Upsilon^{\max}_{R_1, y_l} & \ldots & \Upsilon^{\max}_{R_m, y_l} \end{bmatrix} \tag{57}$$

The proposed formulation is also useful in the area of reliability analysis. The analytical approximation of the response, given by Eq. (52), can be used to determine the probability that the response variable remains below a given threshold at a given time t. Estimating this probability, which can be done using simulation techniques in combination with the approximate system response, is fairly economical from a computational standpoint. In this way, the formulation allows one not only to perform a sensitivity analysis but also to study the system performance from a reliability viewpoint.[23] Finally, other useful applications of this formulation are in connection with structural optimization and fuzzy analysis.[24]

Numerical Examples

To illustrate the applicability of the method, two examples are presented. In the first example, a simple three-story, two-bay building, shown in Fig. 1, is considered. The values of the various parameters describing the structure and loading are set as follows: Elastic modulus for all elements is equal to 2×10^9 kg/m^2; rectangular cross section for all elements with nominal height is equal to 0.6 m and nominal width is equal to 0.3 m; and total weight on the first and second floor is equal to 3.5×10^5 kg and on the third floor is equal

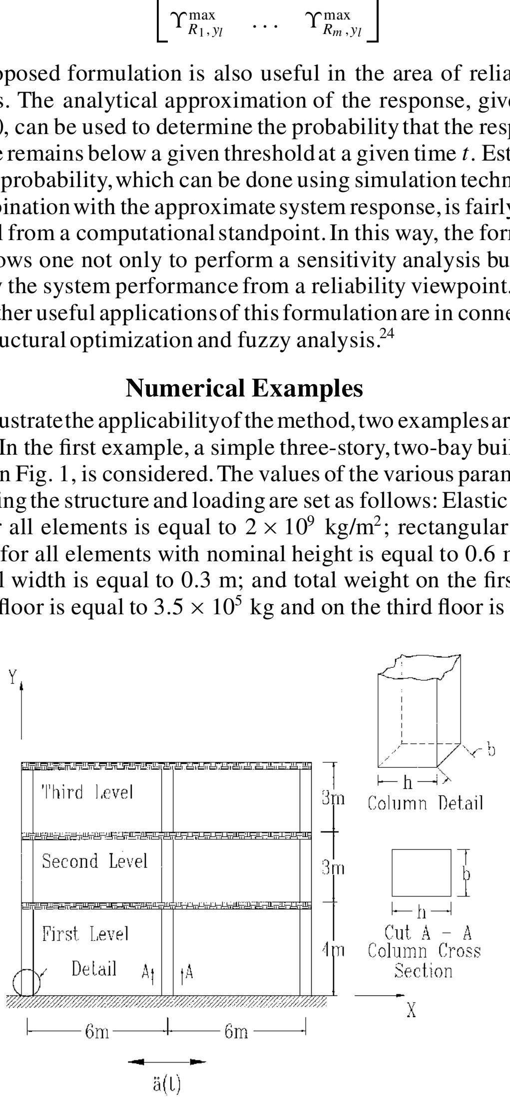

Fig. 1 Three-story, two-bay building.

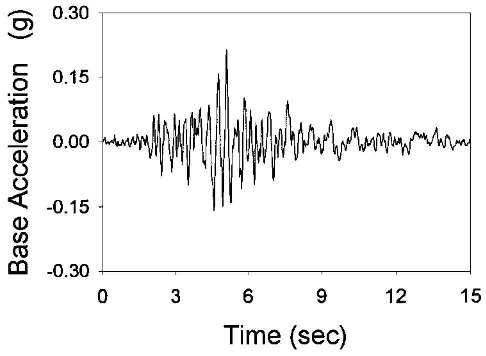

Fig. 2 Base excitation $\ddot{a}(t)$.

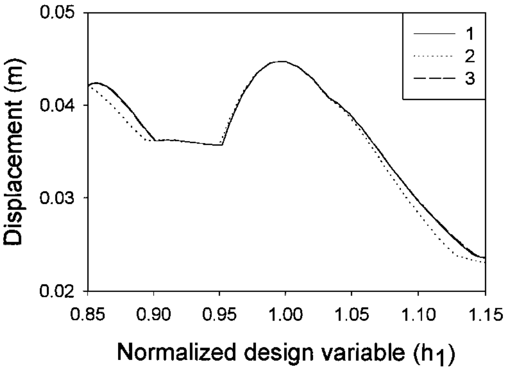

Fig. 3 Maximum horizontal displacement at the top of the building as a function of the design variable h_1: 1, exact response; 2, first-order expansion response; and 3, second-order expansion response.

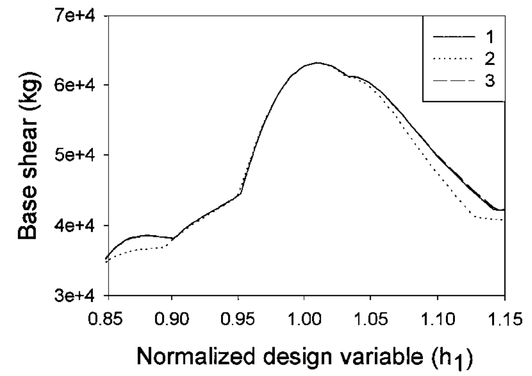

Fig. 4 Maximum base shear as a function of the design variable h_1: 1, exact response; 2, first-order expansion response; and 3, second-order expansion response.

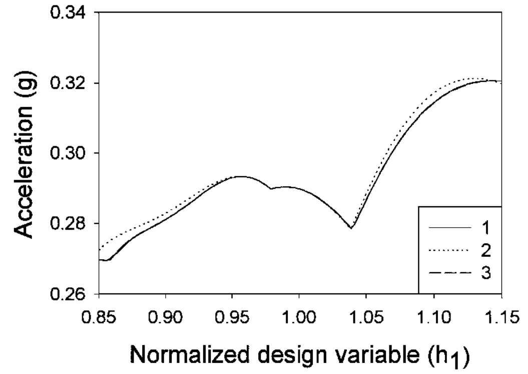

Fig. 5 Maximum absolute acceleration at the top of the building as a function of the design variable h_1: 1, exact response; 2, first-order expansion response; and 3, second-order expansion response.

to 2.8×10^5 kg. To complete the formulation of the system, some amount of damping is added to the model. The system is subjected to a base acceleration $\ddot{a}(t)$, shown in Fig. 2. The design variables are the dimensions of the cross section of the column elements in which the height is denoted by h and the width by b. Three sets of elements are considered: columns corresponding to the first level with design variables b_1 and h_1, columns corresponding to the second level with design variables b_2 and h_2, and columns corresponding to the third level with design variables b_3 and h_3. Therefore, the vector of design variables is given by

$$\{y\}^t = \langle b_1 \quad h_1 \quad b_2 \quad h_2 \quad b_3 \quad h_3 \rangle \tag{58}$$

The response functions to be considered for the sensitivity analysis are the following: the maximum horizontal displacement at the top of the building, R_1; the maximum base shear, R_2; the maximum story drift at the third floor (relative displacement between the second and third floors), R_3; and the maximum absolute acceleration at the top of the building, R_4. Expansions of first and second order in terms of intermediate design variables are used for the approximation of the intermediate response functions (modal energies and eigenvectors). In this example problem, the spectral properties of the system depend on the value of its stiffness matrix. On the other hand, the stiffness coefficients are linear functions of the moments of inertia, and the eigenvectors are in general smooth functions of these variables. Therefore, a good approximation for the modal energies [see Eq. (19)] can be obtained using the moments of inertia as intermediate design variables. For the transient dynamic analysis, all modes are retained due to the simplicity of the model.

One way to evaluate the behavior of the system is through a graphical representation of peak value responses, that is, the maximum response as a function of the system parameters. To illustrate this point, three peak value responses are presented. Figure 3 shows the maximum horizontal displacement at the top of the building as a function of the height of the cross section of the columns corresponding to the first level (design variable h_1). Similarly, Figs. 4 and 5 show the maximum base shear and the maximum absolute acceleration at the top of the building as a function of the design variable h_1. A parameter variability of 15% with respect to its nominal value is considered in Figs. 3–5, and the value of the design variable is normalized by its nominal value. This level of variability produces a 50% variability in the stiffness of the columns corresponding to the first level. To evaluate the performance of the approximations, validation calculations are also shown in Figs. 3–5. The results of the proposed method, using first- and second-order expansion in the approximation of the intermediate response functions, are compared with those obtained using the exact response. The results show that these approximations give excellent results, even with first-order approximations. For the case of second-order expansions, the solution is almost coincident with the exact response. It is noted that the behavior of the maximum story drift at the third floor, R_3, is similar to R_1 in terms of the accuracy of the approximations, but due to space limitation it is not shown here.

Figures 3–5 can also be used to study the behavior of the maximum response over a wide range of values of a design variable, in this case h_1. This type of information can be very useful in redesign analysis. Whenever a design does not meet the performance requirements and needs to be modified, the approximate response helps to identify the critical and more influential design variables. Also, as Figs. 3–5 show, great insight into the behavior of the system can be gained using this formulation.

As already mentioned, the sensitivity of the system responses can also be illustrated through a sensitivity matrix. In this example problem, the following matrix is defined:

$$S = \begin{bmatrix} \Upsilon^{\max}_{R_1,b_1} & \Upsilon^{\max}_{R_2,b_1} & \Upsilon^{\max}_{R_3,b_1} & \Upsilon^{\max}_{R_4,b_1} \\ \Upsilon^{\max}_{R_1,h_1} & \Upsilon^{\max}_{R_2,h_1} & \Upsilon^{\max}_{R_3,h_1} & \Upsilon^{\max}_{R_4,h_1} \\ \Upsilon^{\max}_{R_1,b_2} & \Upsilon^{\max}_{R_2,b_2} & \Upsilon^{\max}_{R_3,b_2} & \Upsilon^{\max}_{R_4,b_2} \\ \Upsilon^{\max}_{R_1,h_2} & \Upsilon^{\max}_{R_2,h_2} & \Upsilon^{\max}_{R_3,h_2} & \Upsilon^{\max}_{R_4,h_2} \\ \Upsilon^{\max}_{R_1,b_3} & \Upsilon^{\max}_{R_2,b_3} & \Upsilon^{\max}_{R_3,b_3} & \Upsilon^{\max}_{R_4,b_3} \\ \Upsilon^{\max}_{R_1,h_3} & \Upsilon^{\max}_{R_2,h_3} & \Upsilon^{\max}_{R_3,h_3} & \Upsilon^{\max}_{R_4,h_3} \end{bmatrix} \tag{59}$$

where the coefficients of the matrix are defined in Eq. (56) and represent the sensitivity of a response with respect to a system parameter. Equation (60) shows the corresponding sensitivity matrix for a range of variation of 20% of the design variables with respect to their nominal values. A first-order approximation is considered in this case:

$$S = \begin{bmatrix} 0.1046 & 0.1314 & 0.1383 & 0.0151 \\ 0.2643 & 0.2775 & 0.2798 & 0.0566 \\ 0.0105 & 0.0199 & 0.0138 & 0.0083 \\ 0.0648 & 0.1124 & 0.0985 & 0.0276 \\ 0.0086 & 0.0043 & 0.1240 & 0.0083 \\ 0.0261 & 0.0176 & 0.4064 & 0.0265 \end{bmatrix} \tag{60}$$

If each column is normalized with respect to the corresponding maximum value, the following matrix is obtained:

$$S_{\text{normalized}} = \begin{bmatrix} 0.40 & 0.47 & 0.34 & 0.27 \\ 1.00 & 1.00 & 0.69 & 1.00 \\ 0.04 & 0.07 & 0.03 & 0.15 \\ 0.25 & 0.41 & 0.24 & 0.49 \\ 0.03 & 0.02 & 0.31 & 0.15 \\ 0.10 & 0.06 & 1.00 & 0.47 \end{bmatrix} \tag{61}$$

These matrices show the more influential and the less influential design variables, in a given region of the design space, with respect to different response functions. For example, the design variable h_1 is the most significant with respect to the maximum horizontal displacement at the top of the building, R_1. In this case, a coefficient of sensitivity of 26% is obtained. Note that this coefficient of sensitivity is greater than the corresponding parameter variability (20%). Therefore, the height of the cross section of the columns corresponding to the first level shows an important influence on the maximum horizontal displacement at the top of the building. This result is expected because h_1 has a significant influence on the spectral properties of the structural system. In the same manner, the design variable h_3 is the most significant with respect to story drift at the third floor, R_3, whereas the parameter b_2 shows a very little influence on that response. Note that a coefficient of sensitivity of 40% is obtained in the case of the design variable h_3, indicating that the height of the cross section of the columns of the third floor has a significant influence on this response. It is also noted that the maximum coefficients for the response functions R_1, R_2, and R_3 are greater than the corresponding parameter variability. This level of response sensitivity shows that some system parameters can markedly alter the response characteristics of the system. Finally, the sensitivity matrix can also be used to evaluate the degree of functional coupling in the set of design variables. For example, the design variable b_1 shows a strong coupling with respect to the response functions R_1, R_2, and R_3 and a weak coupling with respect to the response R_4. Similar analyses can be performed with other response functions and design variables.

In the second application, a simple representation of a printed wiring board with an electronic component mounted on it is considered. These systems are the basic blocks of any electronic assembly because they provide the support to which individual components (capacitors, heat sinks, transformers, etc.) are attached. During manufacturing and operation, a typical board can be subjected to dynamic loads resulting from a number of factors: accidental misuse; vibrations caused by shipping, handling, and transportation and earthquake-generated excitations; environmental stress screening tests; and in avionics, shocks during launching or maneuvering of a spacecraft. Thus, the need to understand the dynamic behavior of printed wiring boards is crucial to assess the performance and reliability of any electronic assembly.

Figure 6 shows a simple structure that consists of a flexible beam supported at both ends by a rotational and a translational spring. In addition, a rigid component is attached to the beam by a set of four springs. The properties (baseline or nominal values) are as follows: $L = 0.3$ m, $l = 0.06$ m, $\rho_{\text{beam}} = 6000$ kg/m^3, $\rho_{\text{component}} = 9000$ kg/m^3, $(EI)_{\text{beam}} = 4.05$ Nm^2, $K_{TS} = 75{,}000$ N/m, $K_{RS} = 230$ Nm/rad, $K_{TC} = 250{,}000$ N/m, and $K_{RC} = 50$ Nm/rad. In addition, it is assumed that the damping is provided by two dashpots located at each end of the beam and having a constant $c = 15$ N s/m. Dimensions for the beam and component cross sections are shown in Fig. 6. The structure is subjected to a vertical ground acceleration $\ddot{a}(t)$ given by the profile (shock) shown in Fig. 6. Five design variables are considered: the stiffness of the beam EI and the values of the four spring constants (K_{TS}, K_{RS}, K_{TC}, K_{RC}). In addition, two dynamic responses are considered: the curvature at the beam centerpoint, R_1, and the total acceleration experienced by the electronic component, R_2. In this model, the intermediate response functions depend directly on the design variables defined earlier. Thus, the original design variables are taken as the intermediate design variables. The system is modeled using an assembly of 10 beam elements plus linear springs, and the first 10 modes are retained in the transient analysis. The effect of higher-order modes on the solution is negligible for this case.

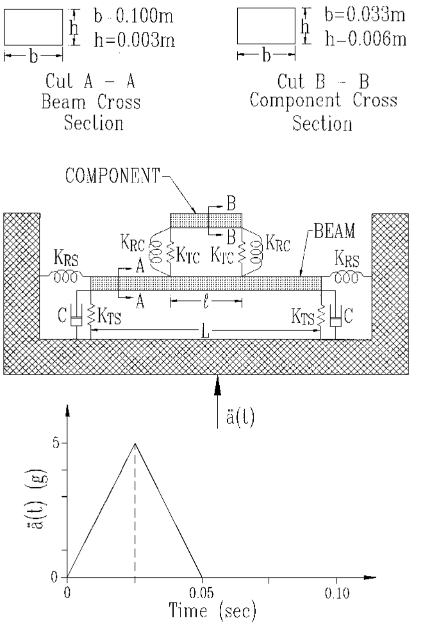

Fig. 6 Simple model of a printed wiring board and shock acceleration profile.

In this application, a sensitivity matrix defined as

$$S = \begin{bmatrix} \Upsilon^{\max}_{R_1,EI} & \Upsilon^{\max}_{R_2,EI} \\ \Upsilon^{\max}_{R_1,K_{TS}} & \Upsilon^{\max}_{R_2,K_{TS}} \\ \Upsilon^{\max}_{R_1,K_{RS}} & \Upsilon^{\max}_{R_2,K_{RS}} \\ \Upsilon^{\max}_{R_1,K_{TC}} & \Upsilon^{\max}_{R_2,K_{TC}} \\ \Upsilon^{\max}_{R_1,K_{RC}} & \Upsilon^{\max}_{R_2,K_{RC}} \end{bmatrix} \tag{62}$$

is used to illustrate the sensitivity of the system for the peak responses of curvature and component acceleration. Equation (63) shows the corresponding sensitivity matrix for a range of variation of 20% of the design variables with respect to their nominal values:

$$S = \begin{bmatrix} 0.1810 & 0.0613 \\ 0.0526 & 0.0527 \\ 0.0670 & 0.0499 \\ 0.0264 & 0.0211 \\ 0.0586 & 0.0243 \end{bmatrix} \tag{63}$$

As in the first example, a normalized sensitivity matrix can be obtained by dividing each column by the corresponding maximum value. This normalization gives

$$S_{\text{normalized}} = \begin{bmatrix} 1.00 & 1.00 \\ 0.29 & 0.86 \\ 0.37 & 0.81 \\ 0.15 & 0.34 \\ 0.32 & 0.40 \end{bmatrix} \tag{64}$$

It is clear that the design variable EI (beam stiffness) has the greatest influence on the value of the curvature, whereas the design variable K_{TC} has minimal influence on the curvature. This is to be expected from an intuitive point of view because the stiffness of the vertical beam supports K_{TS} is rather high compared to the beam vertical stiffness. At the same time, the beam stiffness has the strongest influence on the component acceleration, although it is not as pronounced as in the case of the curvature response. These results also show how misleading it could be to judge the performance of an electronic structure based only on information from its nominal response. Some parameters such as the rigidity of the supports or the flexibility of a chip-module connection are notoriously difficult to estimate, and they can have a significant effect on the system response. Therefore, the ability to quantify the influence of uncertainties in some design variables is crucial. In the context of electronic systems, these results are also useful at the testing stage because they can help to identify the variables that should be determined with more accuracy, due to their impact on the overall response of the structure. Finally, as in the first example, validation calculations performed showed that the approximations considered in the proposed formulation give excellent results.

Conclusions

A sensitivity analysis in which the sensitivity of the system is evaluated by considering the behavior of the system response when the design parameters vary in a region of the design space has been described. The method is based on the approximation of response functions, and it permits the analyst to quantify the response sensitivity by means of coefficients of sensitivity. At the same time, the method allows one to identify the more influential design variables from a global viewpoint, within a region of interest. The degree of functional coupling in the set of design variables can also be obtained directly through this methodology. The present formulation can handle, in an approximate manner, general nonlinear relationships between system parameters and transient response functions. The method uses the modal solution of the equation of motion and makes extensive use of the concepts of intermediate response quantities and intermediate design variables. These concepts are introduced to enhance the approximation of the response functions and at the same time allow an efficient numerical implementation of the method. Validation calculations show that the results from the method agree well with those obtained by using the exact responses, that is, the approximations can accurately capture the nonlinear dependence of the dynamic structural responses on the design variables and system parameters.

Very frequently, sensitivity is estimated using average or nominal responses, that is, the response of the system using average values for the system parameters. The proposed approach takes into account, explicitly, that some parameters are usually estimated with a margin of error that can be significant. Great insight into the behavior of the system response can be gained using the proposed sensitivity approach. Also, it provides valuable information for rational decision making in the design and analysis of complex structural systems.

Acknowledgment

The research reported here was supported in part by the Comisión Nacional de Investigación Científica y Tecnológica under Grant 1970068.

References

[1] Frank, P. M., *Introduction to System Sensitivity Theory*, Academic, New York, 1978, pp. 1–58.

[2] Arora, S. J., and Huag, E. J., "Methods of Design Sensitivity Analysis in Structural Optimization," *AIAA Journal*, Vol. 17, No. 9, 1979, pp. 970–974.

[3] Adelman, H. M., and Haftka, R. T., "Sensitivity Analysis for Discrete Structural Systems," *AIAA Journal*, Vol. 24, No. 5, 1989, pp. 823–831.

[4] Dems, K., and Morz, Z., "Variational Approach by Means of an Adjoint System to Structural Optimization and Sensitivity Analysis," *International Journal of Solids and Structures*, Vol. 20, No. 6, 1984, pp. 527–552.

[5] Dems, K., Mroz, Z., and Szaleg, D., "Optimal Design of Rib-Stiffeners in Disks and Plates," *International Journal of Solids and Structures*, Vol. 28, No. 1, 1989, pp. 973–998.

[6] Cheng, K. T., and Olhoff, N., "Regularized Formulation for Optimal Design of Axisymmetric Plates," *International Journal of Solids and Structures*, Vol. 18, No. 2, 1982, pp. 305–323.

[7] Haug, E. J., and Rousselet, B., "Design Sensitivity Analysis in Structural Mechanics. I. Static Response Variations," *Journal of Structural Mechanics*, Vol. 8, No. 1, 1980, pp. 17–41.

[8] Choi, K. K., and Haug, E. J., "Shape Design Sensitivity Analysis of Elastic Structures," *Journal of Structural Mechanics*, Vol. 11, No. 2, 1983, pp. 231–269.

[9] Dopker, B., Choi, K. K., and Benedict, L., "Shape Design Sensitivity Analysis of Structures Containing Arches," *Computers and Structures*, Vol. 28, No. 1, 1988, pp. 1–13.

[10] Schmit, L. A., and Farshi, B., "Some Approximations Concepts for Efficient Structural Synthesis," *AIAA Journal*, Vol. 12, No. 5, 1974, pp. 692–699.

[11] Vanderplaats, G. N., and Salajegheh, E., "A New Approximation Method for Stress Constraints in Structural Synthesis," *AIAA Journal*, Vol. 27, No. 3, 1989, pp. 352–358.

[12] Thomas, H. L., Sepulveda, A. E., and Schmit, L. A., "Improved Approximations for Control Augmented Structural Optimization," *AIAA Journal*, Vol. 30, No. 1, 1992, pp. 171–179.

[13] Nelson, R. B., "Simplified Calculation of Eigenvector Derivatives," *AIAA Journal*, Vol. 14, No. 9, 1976, pp. 1201–1205.

[14] Morz, Z., Kamat, M. P., and Plaut, R. H., "Sensitivity Analysis and Optimal Design of Nonlinear Beams and Plates," *Journal of Structural Mechanics*, Vol. 13, No. 3, 1985, pp. 245–266.

[15] Ojalvo, I. U., "Efficient Computation of Mode-Shape Derivatives," *Proceedings of the AIAA/ASME/ASCE/AHS/ASC 27th Structures, Structural Dynamics, and Materials Conference* (San Antonio, TX), Vol. 2, AIAA, Washington, DC, 1986, pp. 242–247.

[16] Haftka, R. T., and Morz, Z., "First- and Second-Order Sensitivity Analysis of Nonlinear Structures," *AIAA Journal*, Vol. 24, No. 7, 1986, pp. 1187–1192.

[17] Meric, R. A., "Shape Optimization and Identification of Solid Geometries Considering Discontinuities," *Journal of Heat Transfer*, Vol. 110, No. 3, 1988, pp. 544–550.

[18] Tortorelli, D. A., Lu, C. Y., and Haber, R. B., "Design Sensitivity Analysis for Elastodynamic Systems," *Mechanics of Structures and Machines*, Vol. 18, No. 1, 1990, pp. 77–105.

[19] Sepulveda, A. E., and Thomas, H. L., "Improved Transient Response Approximation for General Damped Systems," *AIAA Journal*, Vol. 34, No. 6, 1996, pp. 1261–1269.

[20] Barthelemy, J. F. M., and Haftka, R. T., "Approximation Concepts for Optimum Structural Design—A Review," *Structural Optimization*, Vol. 5, No. 3, 1993, pp. 129–144.

[21] Iwan, W. D., and Jensen, H., "On the Dynamic Response of Continuous Systems Including Model Uncertainty," *Journal of Applied Mechanics*, Vol. 60, No. 2, 1993, pp. 484–490.

[22] Beck, J. L., and Katafygiotis, L. S., "Treating Model Uncertainties in Structural Dynamics," *Proceedings of 9th World Conference on Earthquake Engineering* (Tokyo–Kyoto, Japan), Japan Association for Earthquake Disaster Prevention, Tokyo, Japan, 1988, pp. 301–306.

[23] Sepulveda, A. E., and Jensen, H., "Comparison of Local and Global Approximations for Reliability Estimation," *AIAA Journal*, Vol. 34, No. 12, 1996, pp. 2162–2170.

[24] Jensen, H., and Sepulveda, A. E., "Approximation Concepts in Design with Fuzzy Representation," *Proceedings of the AIAA/ASME/ASCE/AHS/ASC 37th Structures, Structural Dynamics, and Materials Conference* (Salt Lake City, UT), AIAA, Reston, VA, 1996, pp. 1908–1919.

A. Berman
Associate Editor